Density Matrix and Dynamical aspects of Quantum Mechanics with Fundamental Length

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PACS: 03.65; 05.30

Keywords: fundamental length,deformation,density matrix, Liouville equation

Abstract

In this paper Quantum Mechanics with Fundamental Length is built as a deformation of Quantum Mechanics. To this aim an approach is used which does not take into account commutator deformation as usually it is done, but density matrix deformation. The corresponding deformed density matrix, which is called density pro-matrix is given explicitly. It properties have been investigated as well as some dynamical aspects of the theory. In particular, the deformation of Liouville equation is analyzed in detail. It was shown that Liouville equation in Quantum Mechanics appears as a low energy limit of deformed Liouville equation in Quantum Mechanics with Fundamental Length. Some implications of obtained results are presented as well as their application to the calculation of black hole entropy.

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1 Introduction

The last decade quite intensively was investigated Quantum Mechanics with Fundamental Length (QMFL). The main motivation for these investigations is the description of quantum gravity effects which become considerable only at the Planck scale. To research nature at this scale it is necessary to take into account the concept of minimal length, as it was shown in [1]. The concept of fundamental length was introduced in papers on String Theory using General Uncertainty Relations (GUR) [2].

In this paper QMFL is analyzed from the measurement procedure point of view. It was shown that if the generally accepted measurement rules are used, then density matrix should be deformed or in other words, it should be changed by it progenetrix (density pro-matrix) with $Sp[\rho] < 1$, which appears when Quantum Mechanics (QM) is deformed. As deformation parameter was chosen the quantity $\beta = l_{min}^2/x^2$, where l_{min} is the minimal length and x is the scale. This deformation conducts to QMFL. In this paper that deformation is described explicitly. It was shown that QM appears at the grained scale limit (low energy scale). In such a way the paradigm of inflationary model contains two different (non equivalent) versions of Quantum Mechanics: the first one describes nature at the Planck scale (QMFL) as well as the second one is obtained as the limit when we come back from Planck scale to low energy one (QM). From the given below arguments we conclude that some well-defined concepts in QM (for example, pure state, zero entropy and others) appear only in the low energy limit.

In this paper dynamical aspects of QMFL have been analyzed. A prototype of the Liouville equation has been obtained. It was shown that Liouville equation appears in the low energy limit. The interpretation of established facts is discussed as well as some implications of obtained results. In particular, for the information paradox in black holes. Our approach differs from others since we have considered the deformation of density matrix. At the same time, in other approaches the deformation of commutators has been considered. This paper has to be considered as the logical continuation of [3].

2 Analogue of density Matrix in QMFL

In the last 15 years a lot of papers were issued in which authors, using different methods as: String Theory [2], Gravitation [4], Quantum Theory

of Black Holes [5] and others [6] shown that Heisenberg Uncertainty Relations (UR) [7] should be modified. In particular, a high energy addition has to appear

$$\triangle x \geq \frac{\hbar}{\triangle p} + \alpha L_p^2 \frac{\triangle p}{\hbar}. \tag{1}$$

Where L_p is the Planck length: $L_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 1,6\ 10^{-35}m$ and $\alpha > 0$ is a constant. In paper [4] it was shown that this constant can be chosen equal to 1. However, here we will use α as an arbitrary constant without giving it any concrete value. The inequality (1) is quadratic with respect to $\triangle p$

$$\alpha L_p^2 (\triangle p)^2 - \hbar \triangle x \triangle p + \hbar^2 \leq 0, \tag{2}$$

and from it follows the fundamental length is

$$\triangle x_{min} = 2\sqrt{\alpha} L_p. \tag{3}$$

Since further we are going to base only on the existence of fundamental length, it is necessary to point out this fact was established not only from GUR. For instance, in [8], [9] using an ideal experiment dealing with gravitational field it was obtained the lower bound on limit length, which was improved in [10] without GUR to an estimate of the type $\sim L_p$. Furthermore it is necessary again to remember the review [1], in which it was emphasized that Quantum Gravity, investigated using different approaches, necessarily conducts to the concept of fundamental length.
Let's consider in some detail the equation (3). Squaring it left and right side, we obtain

$$(\overline{\Delta \widehat{X}^2}) \geq 4\alpha L_p^2, \tag{4}$$

or in terms of density matrix

$$Sp[\rho(\widehat{X} - Sp(\rho\widehat{X})^2)] = Sp[(\rho\widehat{X}^2) - Sp^2(\rho\widehat{X})] \geq 4\alpha L_p^2 > 0. \tag{5}$$

where $\widehat{X}$ is the coordinate operator. Expression (5) gives the measuring rule used in QM. However, in the case considered here, in comparison with QM, the right part of (5) cannot be done arbitrarily near to zero since it is limited by $l_{min}^2 > 0$, where due to GUR $l_{min} \sim L_p$.

Apparently, this may be due to the fact that QMFL with GUR (1) is unitary non-equivalent to QM with UR. Actually, in QM the left-hand side of (9) can be chosen arbitrary closed to zero, whereas in QMFL this

is impossible. But if two theories are unitary equivalent then, the form of their spurs should be retained. Besides, a more important aspect is contributing to unitary non-equivalence of these two theories: QMFL contains three fundamental constants (independent parameters) G, c and $\hbar$, whereas QM contains only one: $\hbar$. Within an inflationary model (see [9]), QM is the low-energy limit of QMFL (QMFL turns to QM) for the expansion of the Universe. In this case, the second term in the right-hand side of (1) vanishes and GUR turn to UR. A natural way for studying QMFL is to consider this theory as a deformation of QM, turning to QM at the low energy limit (during the expansion of the Universe after the Big Bang). We will consider precisely this option. However differing from author of papers [5] and others, we do not deform commutators, but density matrix, leaving at the same time the fundamental quantum-mechanical measuring rule (5) without changes. Here the following question may be formulated: how should be deformed density matrix conserving quantum-mechanical measuring rules in order to obtain self-consistent measuring procedure in QMFL? For answering to the question we will use the R-procedure. For starting let us to consider R-procedure both at the Planck's energy scale and at the low-energy one. At the Planck's scale $a \approx il_{min}$ or $a \sim iL_p$, where i is a small quantity. Further a tends to infinity and we obtain for density matrix

$$Sp[\rho a^2] - Sp[\rho a]Sp[\rho a] \simeq l_{min}^2 \ \ or \ \ Sp[\rho] - Sp^2[\rho] \simeq l_{min}^2/a^2.$$

Therefore:

1. When $a < \infty$, $Sp[\rho] = Sp[\rho(a)]$ and $Sp[\rho] - Sp^2[\rho] > 0$. Then, $Sp[\rho] < 1$ that corresponds to the QMFL case.

2. When $a = \infty$, $Sp[\rho]$ does not depend on a and $Sp[\rho] - Sp^2[\rho] \to 0$. Then, $Sp[\rho] = 1$ that corresponds to the QM case.

How should be points 1 and 2 interpreted? How does analysis above-given agree to the main result from [26] [1]? It is in full agreement. Indeed, when state-vector reduction (R-procedure) takes place in QM then, always an eigenstate (value) is chosen exactly. In other words, the probability is equal to 1. As it was pointed out in the above-mentioned point 1 the situation

[1]"... there cannot be any physical state which is a position eigenstate since a eigenstate would of course have zero uncertainty in position"

changes when we consider QMFL: it is impossible to measure coordinates exactly since it never will be absolutely reliable. We obtain in all cases a probability less than 1 ($Sp[\rho] = p < 1$). In other words, any R-procedure in QMFL leads to an eigenvalue, but only with a probability less than 1. This probability is as near to 1 as far the difference between measuring scale a and l_{min} is growing, or in other words, when the second term in (1) becomes insignificant and we turn to QM. Here there is not a contradiction with [26]. In QMFL there are not exact coordinate eigenstates (values) as well as there are not pure states. In this paper we do not consider operator properties in QMFL as it was done in [26] but density-matrix properties.

The properties of density matrix in QMFL and QM have to be different. The only reasoning in this case may be as follows: QMFL must differ from QM, but in such a way that in the low-energy limit a density matrix in QMFL must coincide with the density matrix in QM. That is to say, QMFL is a deformation of QM and the parameter of deformation depends on the measuring scale. This means that in QMFL $\rho = \rho(x)$, where x is the scale, and for $x \to \infty$ $\rho(x) \to \widehat{\rho}$, where $\widehat{\rho}$ is the density matrix in QM.

Since on the Planck's scale $Sp[\rho] < 1$, then for such scales $\rho = \rho(x)$, where x is the scale, is not a density matrix as it is generally defined in QM. On Planck's scale we name $\rho(x)$ "density pro-matrix". As follows from the above, the density matrix $\widehat{\rho}$ appears in the limit

$$\lim_{x \to \infty} \rho(x) \to \widehat{\rho}, \tag{6}$$

when GUR (1) turn to UR and QMFL turns to QM.

Thus, on Planck's scale the density matrix is inadequate to obtain all information about the mean values of operators. A "deformed" density matrix (or pro-matrix) $\rho(x)$ with $Sp[\rho] < 1$ has to be introduced because a missing part of information $1 - Sp[\rho]$ is encoded in the quantity l_{min}^2/a^2, whose specific weight decreases as the scale a expressed in units of l_{min} is going up.

3 QMFL as a deformation of QM

Here we are going to describe QMFL as a deformation of QM using the density pro-matrix formalism. In this context density pro-matrix has to be understood as a deformed density matrix in QMFL. As fundamental deformation parameter use $\beta = l_{min}^2/x^2$, where x is the scale and $l_{min} \sim L_p$.

Definition 1.

Any system in QMFL is described by the density pro-matrix $\rho(\beta) = \sum_i \omega_i(\beta)|i><i|$, where

1. $0 < \beta \leq 1/4$;
2. The vectors $|i>$ form a full orthonormal system;
3. $\omega_i(\beta) \geq 0$ and for all i there is a finite limit $\lim\limits_{\beta \to 0} \omega_i(\beta) = \omega_i$;
4. $Sp[\rho(\beta)] = \sum_i \omega_i(\beta) < 1, \sum_i \omega_i = 1$;
5. For any operator B and any β there is a mean operator B, which depends on β: $<B>_\beta = \sum_i \omega_i(\beta) <i|B|i>$.

At last, in order to match our definition with the result of section 2 the next condition has to be fulfilled:

$$Sp[\rho(\beta)] - Sp^2[\rho(\beta)] \approx \beta, \tag{7}$$

from which we can find the meaning of the quantity $Sp[\rho(\beta)]$, which satisfies the condition of definition:

$$Sp[\rho(\beta)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \beta}. \tag{8}$$

From point 5) it follows,that $<1>_\beta = Sp[\rho(\beta)]$. Therefore for any scalar quantity f we have $<f>_\beta = f Sp[\rho(\beta)]$. In particular, the mean value $<[x_\mu, p_\nu]>_\beta$ is equal to

$$<[x_\mu, p_\nu]>_\beta = i\hbar\delta_{\mu,\nu} Sp[\rho(\beta)] = i\hbar\delta_{\mu,\nu}(\frac{1}{2} + \sqrt{\frac{1}{4} - \beta}) \tag{9}$$

We obtain density matrix as the limit $\lim\limits_{\beta \to 0} \rho(\beta) = \rho$. It is evident, that in the limit $\beta \to 0$ we turn to QM. Here we would like to verify, that both two cases described above correspond to the meanings of β. In the first case when β is near to $\frac{1}{4}$. In the second one when it is near to zero.
From the definitions given above it follows that $<(j><j)>_\beta = \omega_j(\beta)$.
From which the condition of completeness on β is $<(\sum_i |i><i|)>_\beta = <1>_\beta = Sp[\rho(\beta)]$. The norm of any vector $|\psi>$, assigned to β can be defined as

$< \psi|\psi >_\beta =< \psi|(\sum_i |i >< i|)_\beta|\psi > =< \psi|(1)_\beta|\psi > =< \psi|\psi > Sp[\rho(\beta)$, where $< \psi|\psi >$ is the norm in QM, or in other words when $\beta \to 0$. By analogy, for probabilistic interpretation the same situation takes place in the described theory, but only changing ρ by $\rho(\beta)$.

Some remarks:

I. The considered above limit covers at the same time Quantum and Classical Mechanics. Indeed, since $\beta \sim L_p^2/x^2 = G\hbar/c^3x^2$, so we obtain:

   a. $(\hbar \neq 0, x \to \infty) \Rightarrow (\beta \to 0)$ for QM;

   b. $(\hbar \to 0, x \to \infty) \Rightarrow (\beta \to 0)$ for Classical Mechanics;

II. In reality the parameter of deformation β should take the meaning $0 < \beta \leq 1$. However, as we can see from (8), and as it was indicated in the section 2, $Sp[\rho(\beta)]$ is well defined only for $0 < \beta \leq 1/4$. Some troubles can appear only for the point with fundamental length, since if $x = 2l_{min}$,then the problem vanishes. At the very point with fundamental length $x = l_{min} \sim L_p$ there is a singularity, which is connected with the appearance of the complex meaning of $Sp[\rho(\beta)]$, or in other words it is connected with the impossibility of diagonalization density pro-matrix at this point over the field of real numbers. For this reason definition 1 at the initial point does not have any sense.

III. We have to consider the question about solutions of (7). For instance, one of the solutions of (7), at least at first order on β is $\rho^*(\beta) = \sum_i \alpha_i exp(-\beta)|i >< i|$, where all $\alpha_i > 0$ do not depend on β and their sum is equal 1. Indeed, we can easy verify that

$$Sp[\rho^*(\beta)] - Sp^2[\rho^*(\beta)] = \beta + O(\beta^2) \tag{10}$$

Here it is necessary to consider that in momentum's representation $\beta = p^2/p_{pl}^2$, where p_{pl} is the Planck momentum. In the case when the quantity $exp(-\beta)$ is present in the matrix elements it can dump out the contribution of large momentum in perturbation theory.

IV. It is clear, that in the proposed above description states, which have a probability equal to 1 (pure state), can appear only in the limit $\beta \to 0$, or when all states $\omega_i(\beta)$ except one of them are equal to zero, or when they tend to zero at this limit.

V. We suppose, that all definitions concerning density matrix can be transfer to the described above deformation of Quantum Mechanics (QMFL) changing the density matrix ρ by the density pro-matrix $\rho(\beta)$ and turning then to the low energy limit $\beta \to 0$. In particular, for statistical entropy we have

$$S_\beta = -Sp[\rho(\beta)\ln(\rho(\beta))]. \tag{11}$$

The quantity S_β, evidently never is equal to zero, since $\ln \rho(\beta) \neq 0$ and, therefore S_β may be equal to zero only at the limit $\beta \to 0$.

1. If we carry out a measurement in a defined scale we cannot consider a density pro-matrix with a precision, which exceed some limit of order $\sim 10^{-66+2n}$, where 10^{-n} is the scale in which the measurement is carried out. In most of the known cases this precision is quite enough for considering density pro-matrix density matrix. However, at the Planck scale, where quantum gravity effects cannot be neglected and energy is of the Planck order the difference between $\rho(\beta)$ and ρ has to be considered.

2. At the Planck scale the notion "Wave function of the Universe", introduced by J.A. Wheeler and B. deWitt [12] does not work and in this case quantum gravity effects can be described only with the help of density pro-matrix $\rho(\beta)$.

3. Since density pro-matrix $\rho(\beta)$ depends on the scale in which the measurement is carried out, then the evolution of the Universe within inflationary model paradigm [11] is not an unitary process, because, otherwise the probability $p_i = \omega_i(\beta)$ would be conserved.

4 Dynamical aspects of QMFL

Let's suppose that in QMFL density pro-matrix has the form $\rho[\beta(t), t]$, or in other words it depends on two parameters: time t and deformation parameter β, which also depends on time $\beta = \beta(t)$. Then we have

$$\rho[\beta(t), t] = \sum_i \omega_i[\beta(t)] |i(t)><i(t)|. \tag{12}$$

We obtain the equation

$$\frac{d\rho[\beta(t),t]}{dt} = \sum_i \frac{d\omega_i[\beta(t)]}{dt} |i(t)><i(t)| - i[H,\rho(\beta)] \qquad (13)$$

This is a prototype of the Liouville equation (deformed Liouville equation in QMFL).

Let's consider some particular cases.

1. First we consider the process of time evolution at low energies, or in other words, when $\beta(0) \approx 0$, $\beta(t) \approx 0$ and $t \to \infty$. Then it is clear that $\omega_i(\beta) \approx \omega_i \approx constant$. The first term in (13) vanishes and we obtain the Liouville equation.

2. We obtain also the Liouville equation when we turn from inflation to big scale. Within the inflationary approach the scale $a \approx e^{Ht}$, where H is the Hubble's constant and t is time. Therefore $\beta \approx e^{-2Ht}$ and when t is quite big $\beta \to 0$. In other words $\omega_i[\beta] \to \omega_i$, the first term in (13) vanishes and we again obtain the Liouville equation.

3. At very early stage of inflationary process or even before it takes place $\omega_i[\beta]$ was not a constant and therefore the first term in (13) should be taking into account. This way we obtain a deviation from the Liouville equation.

4. At last let's consider the case when $\beta(0) \approx 0$, $\beta(t) > 0$ when $t \to \infty$. In this case we are going from low energy scale to high energy one and $\beta(t)$ grows when $t \to \infty$. In this case all terms in (13) are significant and we obtain an addition to the Liouville equation in the form

   $$\sum \frac{d\omega_i[\beta(t)]}{dt} |i(t)><i(t)|.$$

   This case can take place when matter go into a Black Hole and it is moving in the direction of the singularity (to the Planck scale).

5 Analysis of the Information Paradox Problem in Black Holes

Results obtained so far allow us anew to give a meaning to the problem of information loss in a Black Hole [13, 14, 15], at least for Quantum Black

Holes with a big curvature. Indeed, when we deal with Black Holes of this nature quantum effects are considerable at the Planck scale and as it was shown above the initial entropy of matter, absorbed by a Black Hole at this scale cannot be equal to zero. This is in agreement with the R.Myers's results: pure quantum states do not form Black Holes [16]. Due to this result the problem of information loss on Black Holes should be reformulated in other way, since in all papers on information paradox zero entropy at the initial state is compared with nonzero entropy at the final state. Moreover it is necessary to note, that last time in some papers GUR for Black Holes are considered at the very beginning [17]. As a consequence of this approach stable remnants with mass of the order of Planck mass M_{Pl} appear during the evaporation of Black Holes. The last affirmation allows us to conclude that Black Holes should not have evaporated completely. Results given in [18] can be applied to the semi-classical case and are not suitable for considering Quantum Black Holes.

On the other hand, from the results obtained above, at least at the qualitative level, it can be elucidated the answer to the question how could be lost information in big Black Holes, which are formed as the result of a star collapse? Our point of view is closed to the R.Penrose's one [19]. He considers (in opposition to S. Hawking) that information in Black Holes is lost when matter meets a singularity. In our approach information loss takes place in the same form. Indeed, near to the horizon of events an approximately pure state with practically equal to zero initial entropy $S^{in} = -Sp[\rho \ln(\rho)]$, which corresponds to $\beta \to 0$, when approaching to a singularity (in other words is reaching the Planck scale) gives yet non zero entropy $S_{\beta} = -Sp[\rho(\beta) \ln(\rho(\beta))] > 0$, when $\beta > 0$. Therefore entropy increases and information is lost at this Black Hole. We can (at the moment also at the qualitative level) evaluate entropy of Black Holes. Indeed, starting from density matrix for a pure state at the "entry" of a Black Hole $\rho_{in} = \rho_{pure}$ with zero entropy $S^{in} = 0$, we obtain at the singularity in the Black Hole density pro-matrix $\rho_{out} = \rho(\beta) \approx \rho(1/4)$ with entropy $S^{out} = S_{1/4} = -Sp[\rho(1/4) \ln(\rho(1/4)] = -1/2 \ln 1/2 \approx 0.34657$. If we take into account that total entropy of a Black Hole is proportional to quantum area of surface A, measured in Planck units of area L_p^2 [20], we obtain the next value for Quantum entropy of a Black Hole:

$$S_{blackhole} = 0.34657A \tag{14}$$

This value differs from the well-known one given by Bekenstein-Hawking

for Black Hole entropy $S_{blackhole} = \frac{1}{4}A$ [21]. The last value was obtained in the semi-classical approximation. At the present moment quantum corrections to this value are investigated [22]. Our approach based on the quantum nature of Black Holes allows to obtain formula (14) from basic principles. Let us to note here that in the approaches, used up to now to modify Liouville equation due to information paradox [23], the additional member appearing in the right side of $d\rho/dt$, where ρ is density matrix, has the form

$$-\frac{1}{2}\sum_{\alpha\beta\neq 0}(Q^{\beta}Q^{\alpha}\rho + \rho Q^{\beta}Q^{\alpha} - 2Q^{\alpha}\rho Q^{\beta})$$

where Q^{α} is a full orthogonal set of Hermitian matrices with $Q^0 = 1$. In this case either locality or conservation of energy-momentum tensor is broken down. In the offered in this paper approach, the added member in the deformed Liouville equation has a more natural and beautiful form, in our opinion:

$$\sum_i \frac{d\omega_i[\beta(t)]}{dt}|i(t)><i(t)|.$$

All properties of QM are conserved in the limit $\beta \rightarrow 0$, in which the added member vanishes and we obtain Liouville equation.

6 Conclusion

Measurement procedure in QM consists of two parts: measurement rules and measurement operator. In this paper we try to answer to the following question: if measurement rules (5) have not been changed, then how is deformed measurement operator when QM is deformed? It is clear that in this case, since one of the components of the measurement procedure has been deformed (in particular density matrix), then the measurement procedure itself should be changed also. Here it is rightful to formulate the following question: is it correct to use generally accepted measurement rules in Quantum gravity? Usually in Quantum Gravity precisely the generally accepted rules are used [24], but measurement operator is not deformed. However, according to the given above arguments, the measurement operator should be deformed.

As it was noted in [1] all known approaches, used to substantiate Quantum Gravity one way or another conduct to the concept of "fundamental length". Furthermore GUR (1) which also conduct to this concept are well

incorporated within the inflationary model [25]. Therefore to understand nature at the planck scale, leaving apart the concept of fundamental length it seems to be no possible.

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